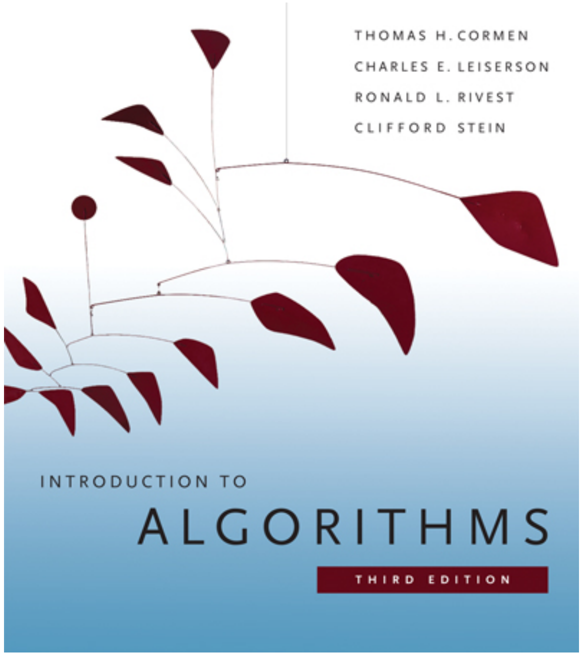
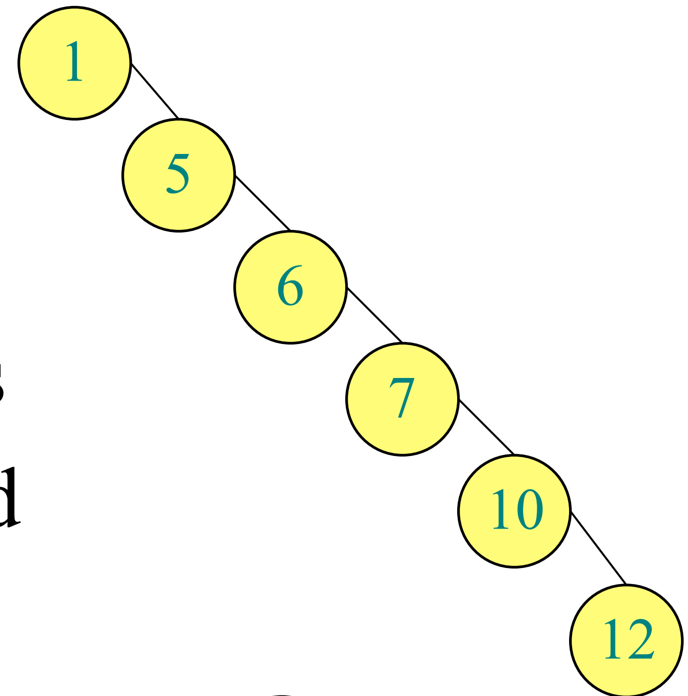
6.006- *Introduction to Algorithms*



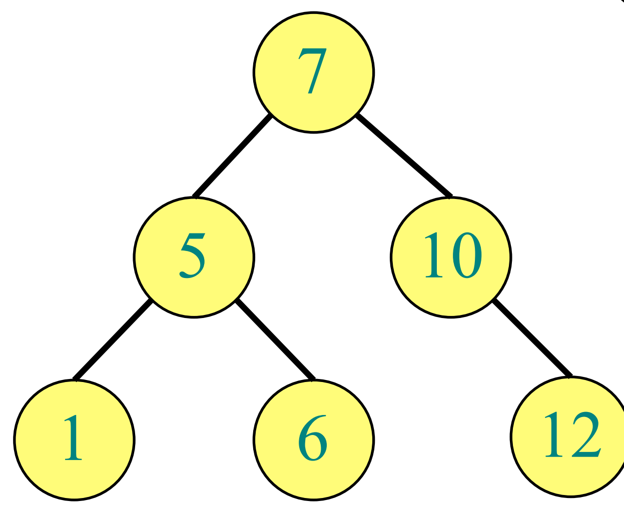
Lecture 4

Prof. Piotr Indyk

Lecture Overview



- Review: Binary Search Trees
- Importance of being balanced
- Balanced BSTs
  - AVL trees
    - definition
    - rotations, insert

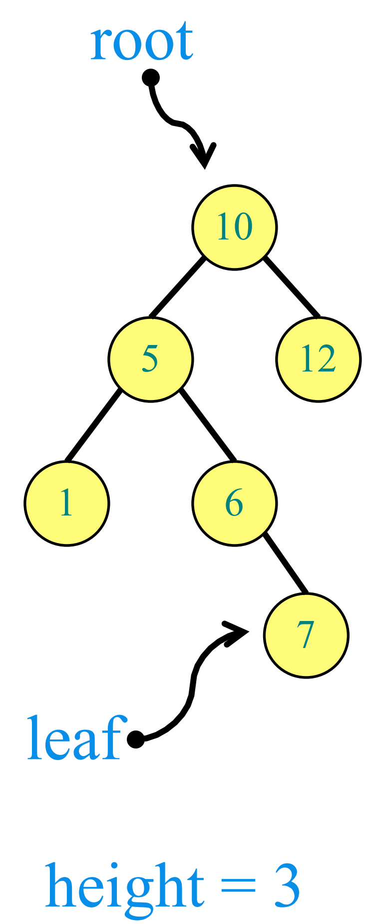


Binary Search Trees (BSTs)

- Each node x has:
  - key[x]
  - Pointers: left[x], right[x], p[x]
- Property: for any node x:
  - For all nodes y in the left subtree of x:

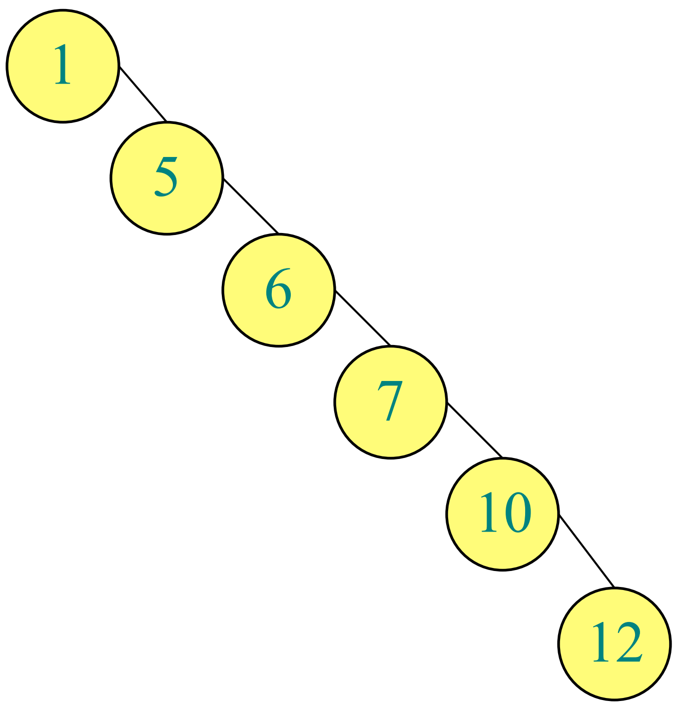
    $key[y] \leq key[x]$
  - For all nodes y in the right subtree of x:

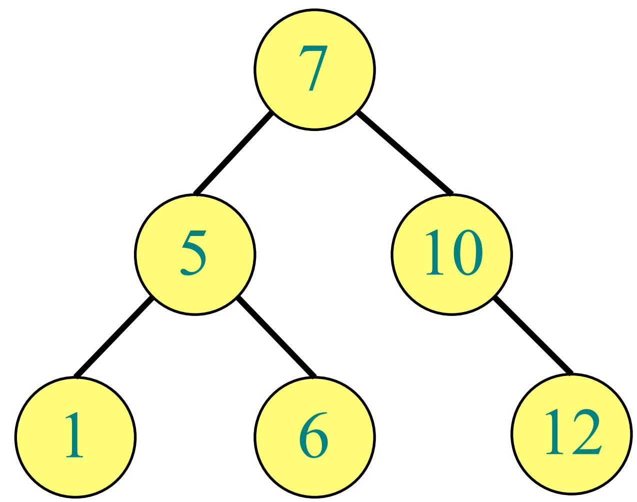
    $key[y] \geq key[x]$



The importance of being balanced

for n nodes:



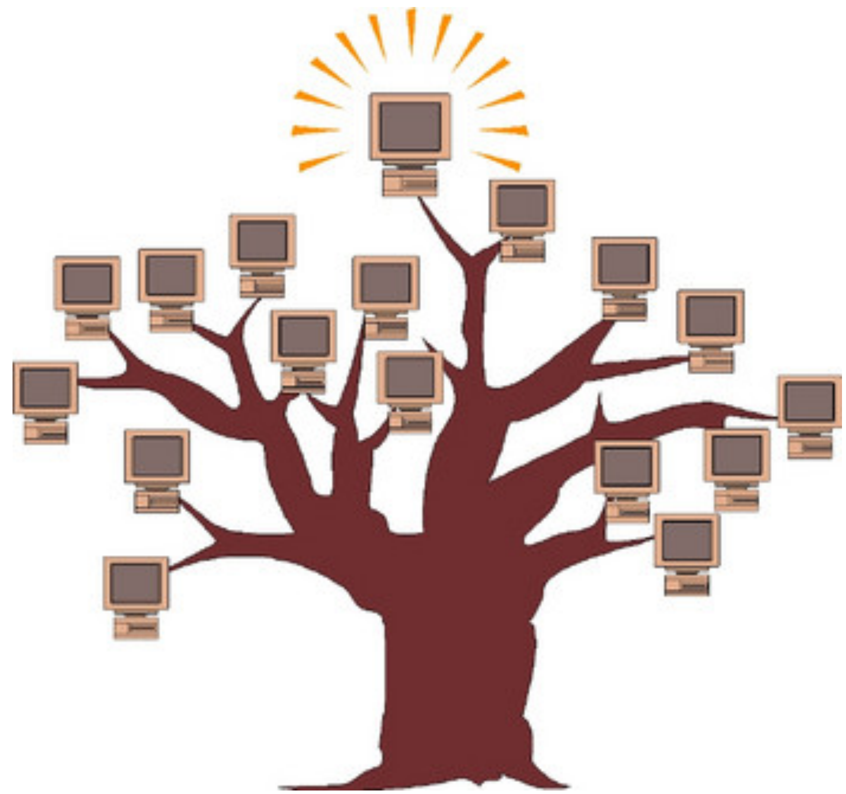


$h = \Theta(\log n)$

$h = \Theta(n)$

Balanced BST Strategy

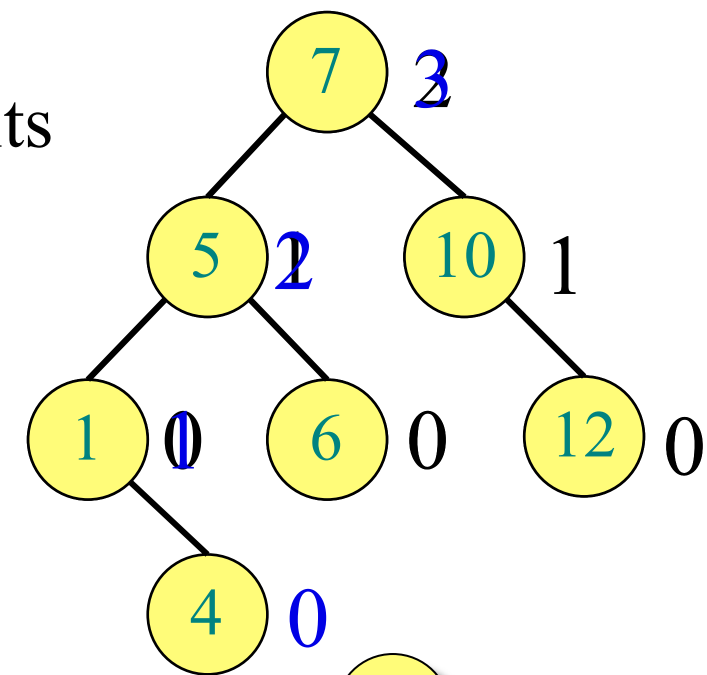
- Augment every node with some data
- Define a local invariant on data
- Show (prove) that invariant guarantees $\Theta(\log n)$ height
- Design algorithms to maintain data and the invariant



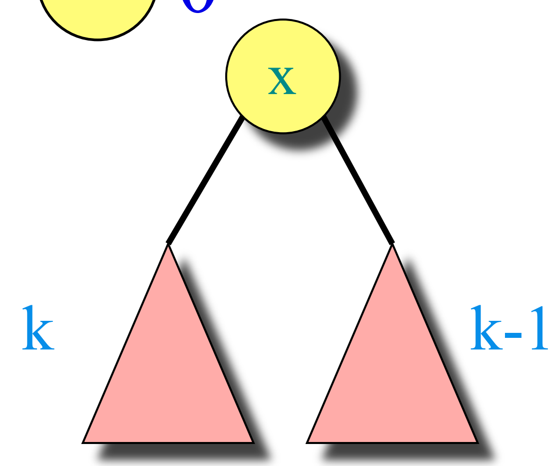
AVL Trees: Definition

[Adelson-Velskii and Landis'62]

- **Data**: for every node, maintain its height ("augmentation")
  - Leaves have height 0
  - NIL has "height" -1



- **Invariant**: for every node x, the heights of its left child and right child differ by at most 1



AVL trees have height $\Theta(\log n)$

Invariant: for every node x, the heights of its left child and right child differ by at most 1

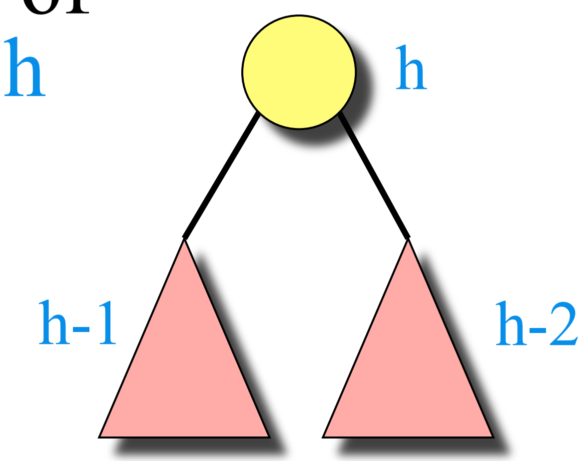
- Let n_h be the minimum number of nodes of an AVL tree of height h
- We have $n_h \geq 1 + n_{h-1} + n_{h-2}$

$$\Rightarrow n_h > 2n_{h-2}$$

$$\Rightarrow n_h > 2^{h/2}$$

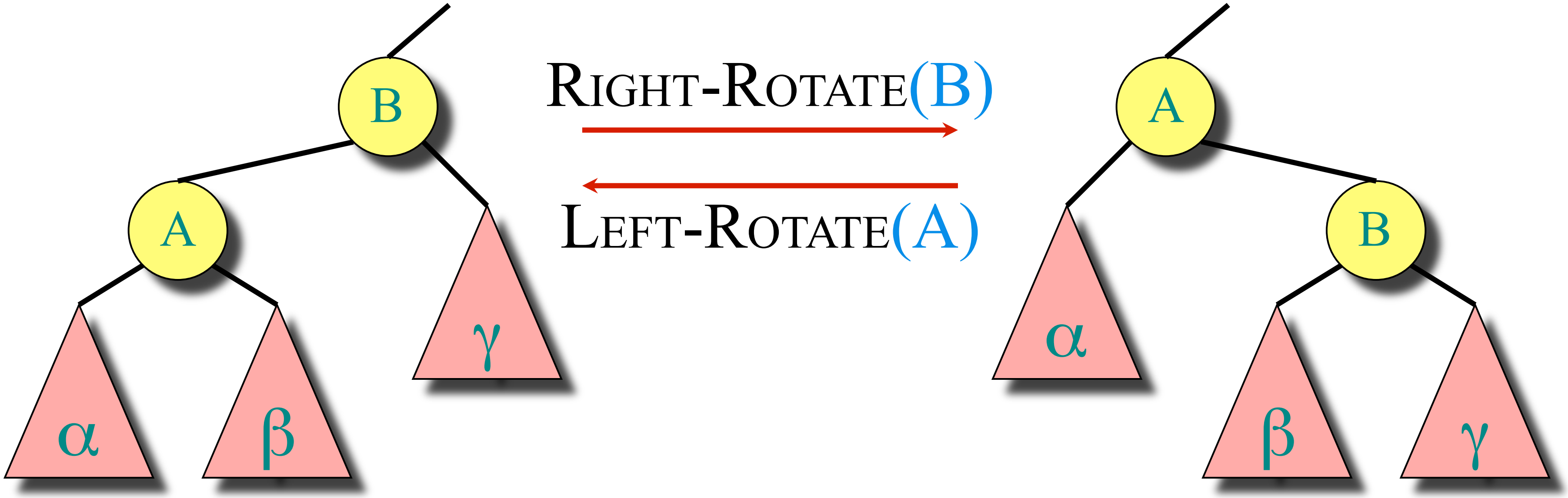
$$\Rightarrow h < 2 \lg n_h$$

- The constant "2" can be improved



How can we maintain the invariant ?

Rotations

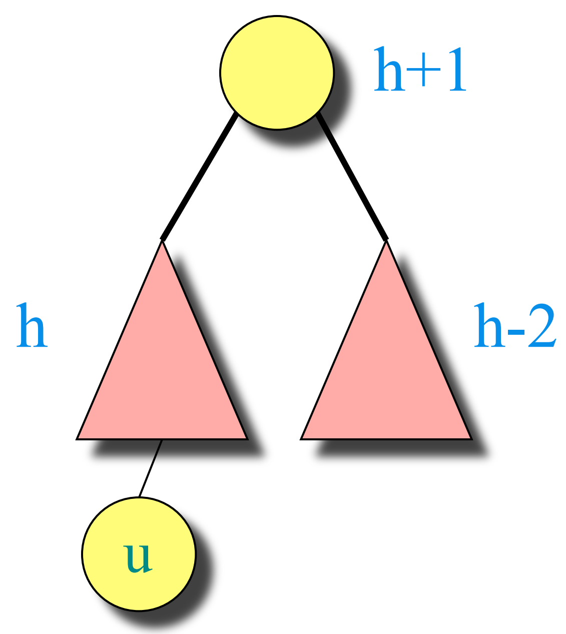


Rotations maintain the inorder ordering of keys:

- $a \in \alpha, b \in \beta, c \in \gamma \;\Rightarrow\; a \le A \le b \le B \le c$.

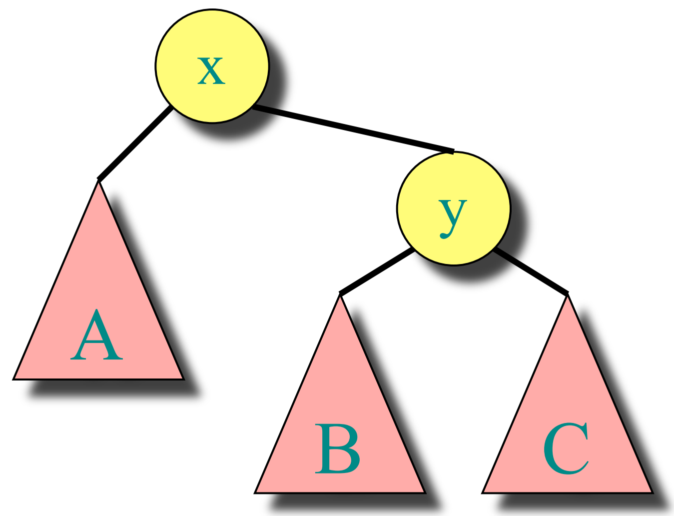
Insertions

- Insert new node u as in the simple BST
  - Can create imbalance
- Work your way up the tree, restoring the balance
- Similar issue/solution when deleting a node

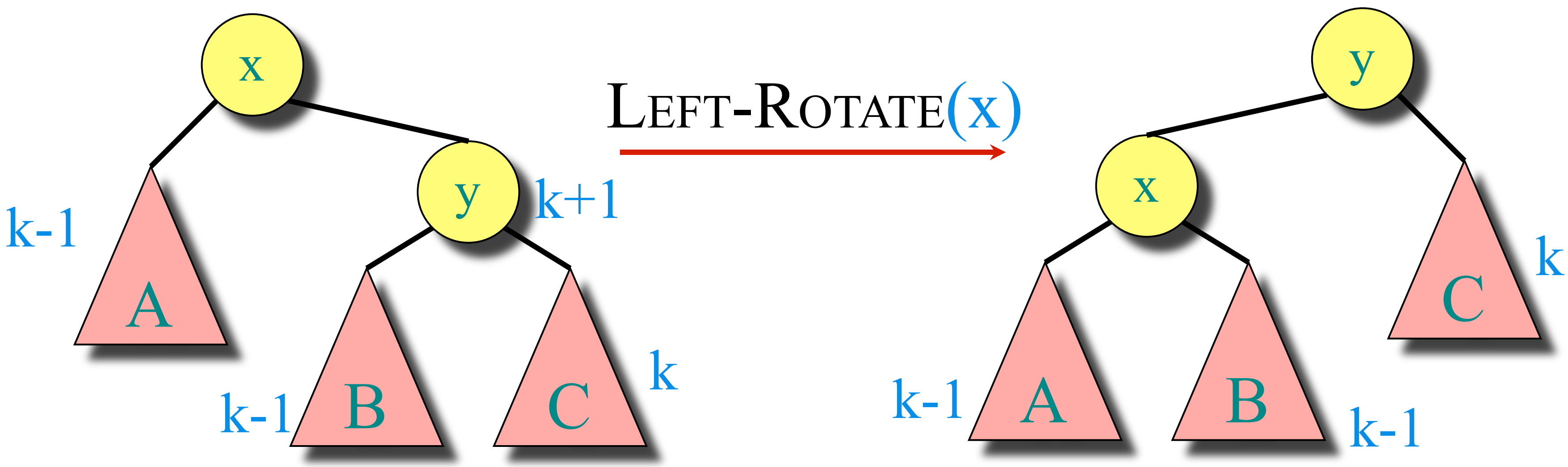


Balancing

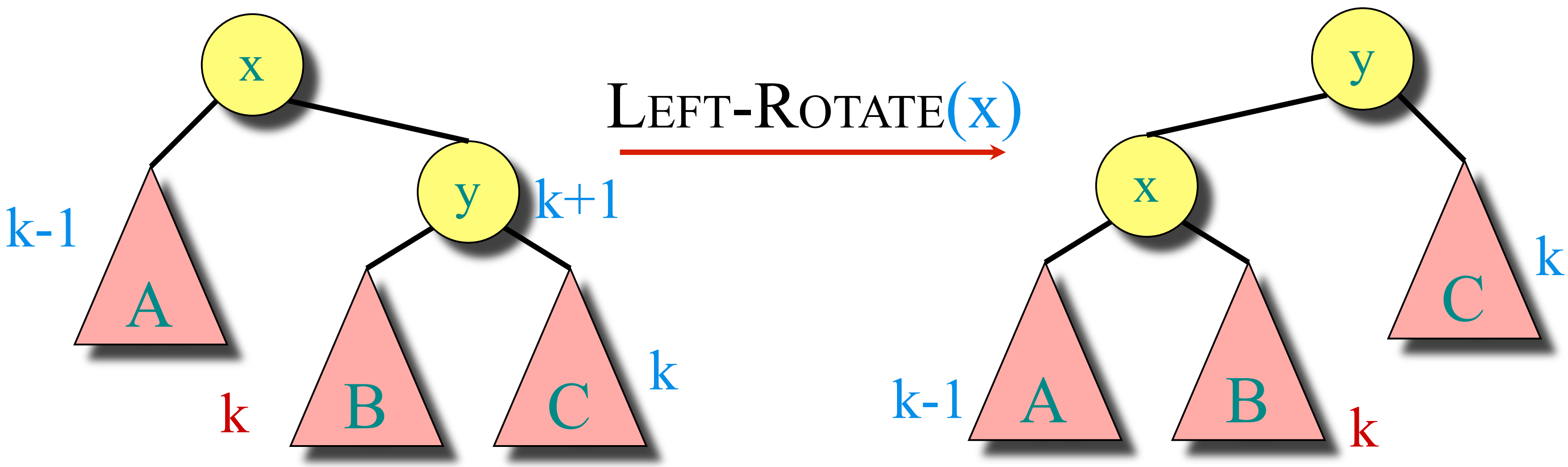
- Let x be the lowest "violating" node
  – We will fix the subtree of x and move up
- Assume the right child of x is deeper than the left child of x (x is "right-heavy")
- Scenarios:
  – Case 1: Right child y of x is right-heavy
  – Case 2: Right child y of x is balanced
  – Case 3: Right child y of x is left-heavy



Case 1: y is right-heavy

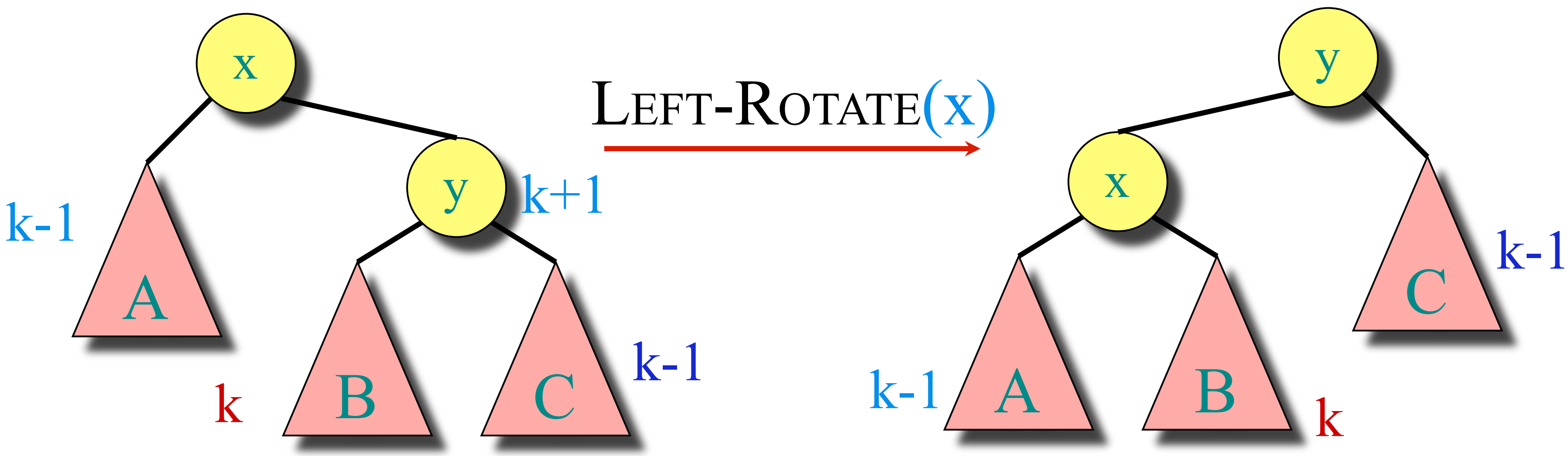


Case 2: y is balanced



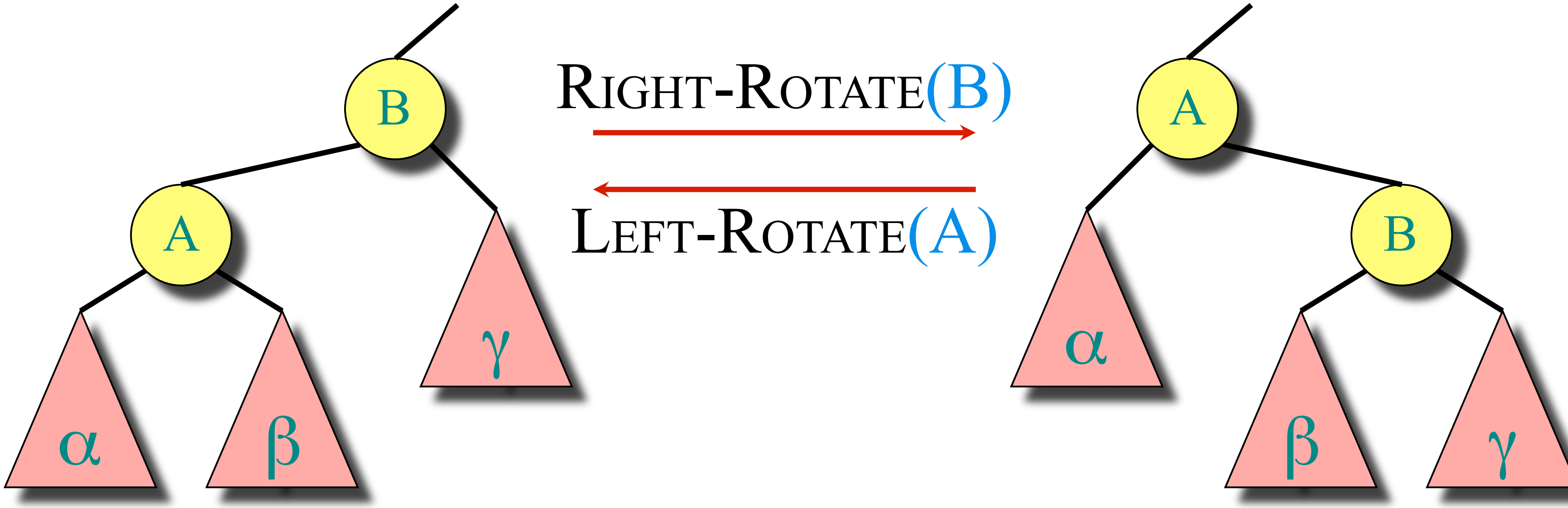
Same as Case 1

Case 3: y is left-heavy



Need to do more …

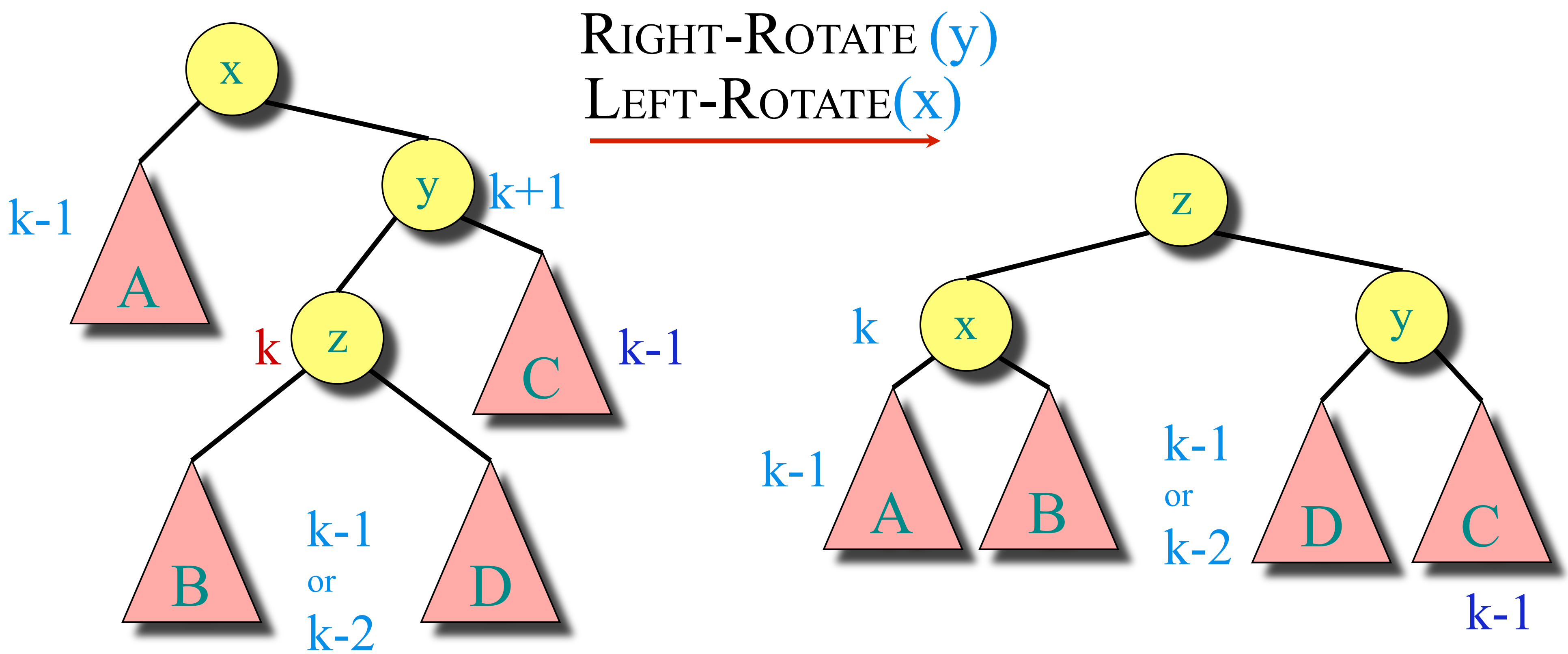
Rotations



Rotations maintain the inorder ordering of keys:

- $a \in \alpha, b \in \beta, c \in \gamma \;\Rightarrow\; a \le A \le b \le B \le c$.

Case 3: y is left-heavy

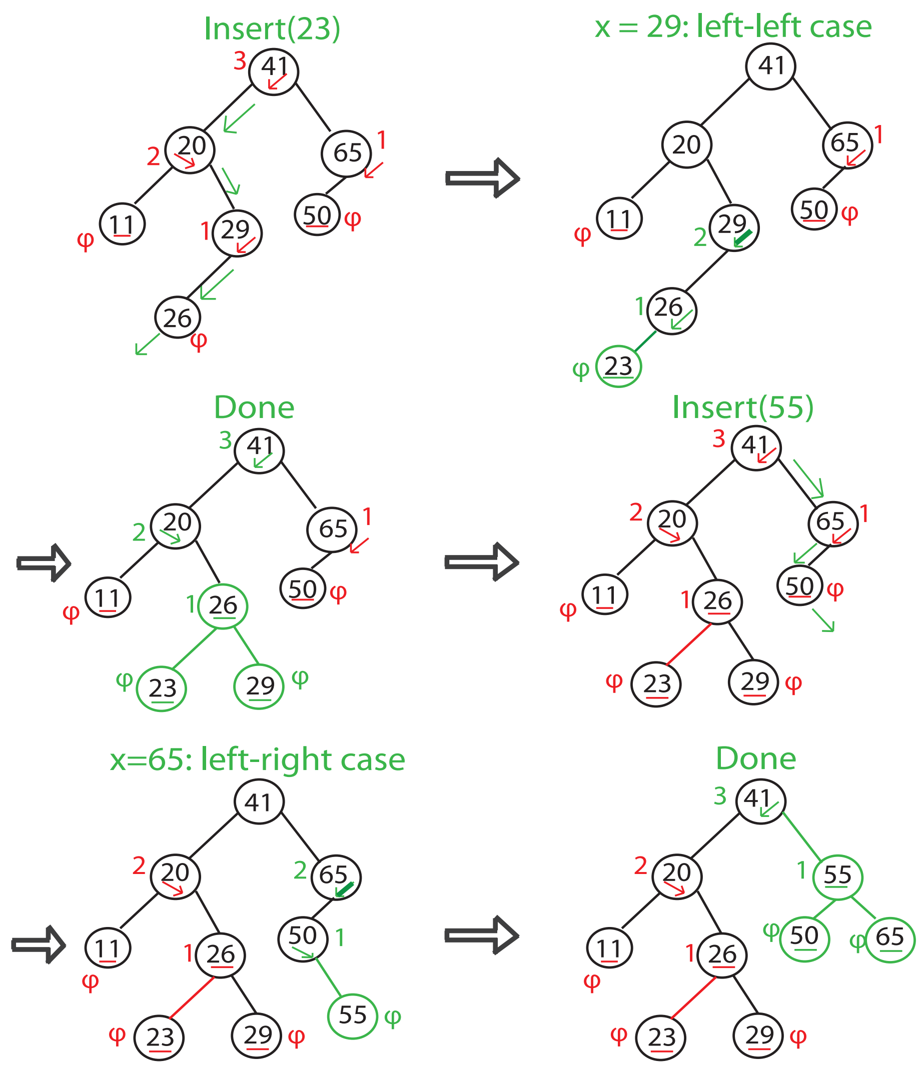


And we are done!

Conclusions

- Can maintain balanced BSTs in O(log n) time per insertion
- Search etc take O(log n) time

Examples of insert/balancing

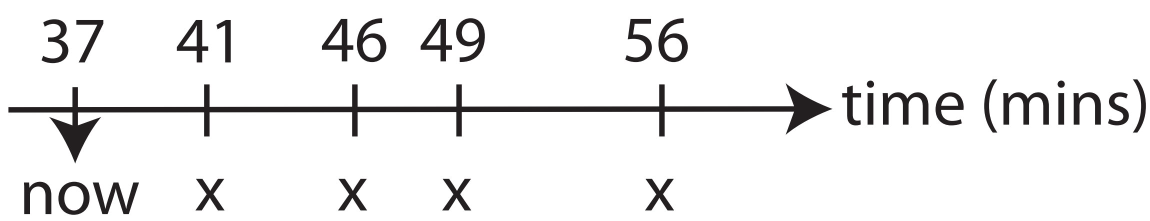


Balanced Search Trees …

- AVL trees (Adelson-Velsii and Landis 1962)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan 1985)
- Scapegoat trees (Galperin and Rivest 1993)
- Treaps (Seidel and Aragon 1996)
- ….

BST for runway reservation system

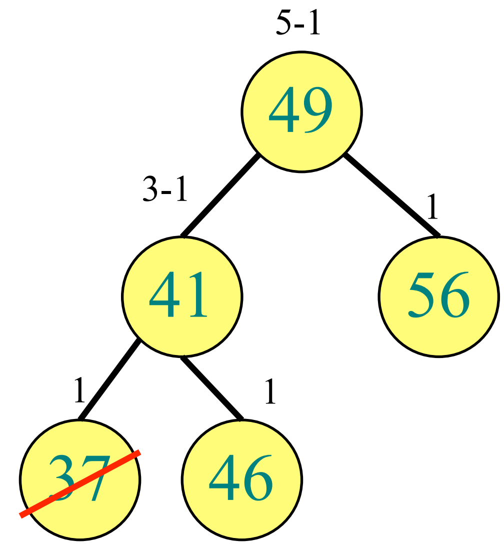
- R = (37, 41, 46, 49, 56) current landing times

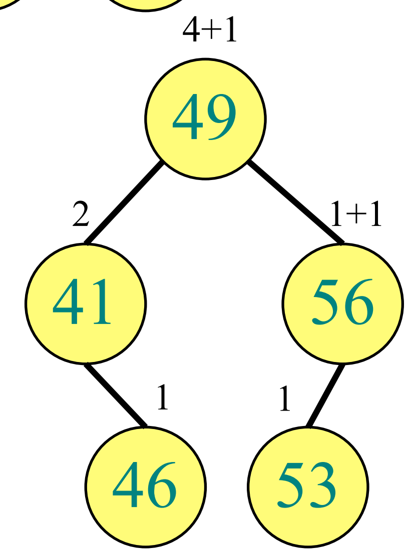


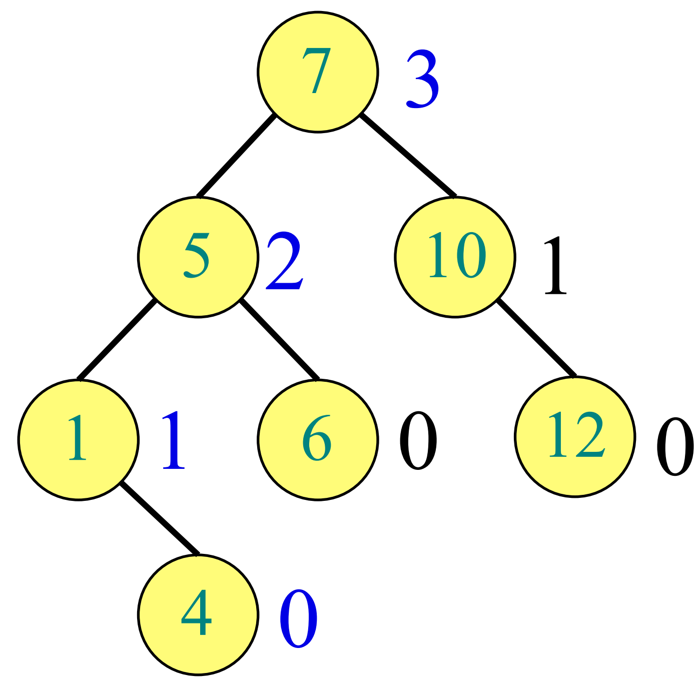
- remove t from the set when a plane lands

  R = (41, 46, 49, 56)

- add new t to the set if no other landings are scheduled within < 3 minutes from t
  - 44 => reject (46 in R)
  - 53 => ok
- delete, insert, conflict checking take O(h), where h is the height of the tree






7
3
5
2
10
1
1
1
6
0
12
0
4
0

And some people like to do nothing